

**Classical Gravitational Back-Reaction**N. C. Tsamis<sup>†</sup>*Institute of Theoretical & Computational Physics, and  
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Gainesville, FL 32611, UNITED STATES.***ABSTRACT**

The quantum gravitational back-reaction on inflation is based on the self-gravitation of infrared gravitons which are ripped out of the vacuum during inflation. The only quantum part of this process is the creation of gravitons; after they have emerged from the vacuum their behaviour is essentially classical. To test the thesis that a sufficiently dense ensemble of classical gravitons can hold the universe together in pure gravity with a positive cosmological constant, we compute the initial value and first time derivative of an invariant measure of the expansion rate for arbitrary classical initial value data. Our result is that the self-gravitation from the kinetic energy of an initial ensemble of gravitons can indeed slow expansion enough to hold the universe together.

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# 1 Introduction

Gravitation plays the dominant role in shaping cosmological evolution. Moreover, a wide variety of observational evidence points to the very early universe having experienced a phase of accelerated expansion, or inflation [1]. During inflation, quantum physics implies the production of real particles out of the vacuum as long as they are effectively massless, possess classically non-conformally invariant free Lagrangians, and have adequately large wavelength. The carrier of the gravitational force, the graviton, is such a particle and inflationary evolution eventually will produce a dense ensemble of infrared gravitons [2].

Gravitation couples to any stress-energy source and the aforementioned quantum induced source of gravitons is no exception. It becomes important, therefore, to study the gravitational response to its presence. Being a universally attractive force, gravity has the potential to alter the inflationary expansion rate and decrease it. This has already been suggested [3] but the supporting perturbative analysis eventually becomes unreliable; the self-gravitation of the infrared gravitons ripped out of the vacuum very slowly but cumulatively increases until perturbation theory breaks down.

That said, the question arises whether we can make any quantitative non-perturbative statements. With this in mind, we note that our physical problem can be stated as the *classical* gravitational back-reaction to a *quantum* induced graviton source; only the particle creation out of the vacuum is a quantum effect. Detailed knowledge of this quantum source would allow similar knowledge of the response to its presence, the latter being determined by the field equations of gravity. However, the non-linearity of the theory is a formidable hindrance both for the description of the graviton source and the response to it.

Nonetheless, it is possible to obtain non-perturbatively some measure of the back-reaction on an initial value surface (IVS) for arbitrary initial value data (IVD). In the real situation the initial value surface would coincide with the beginning of the inflationary era and after considerable time evolution the quantum induced graviton source would slowly but steadily become significant. Even if we lack an analytical form for the source, it *must* correspond to some IVD. Even if full time evolution is beyond our means, we *can* compute the expansion rate and its first time derivative on the IVS.

A physical measure of the back-reaction can be provided by an observable which invariantly determines the expansion rate [4] and which we review in

Section 2. The computation of its initial value and first time derivative for any classical initial value data are presented in Section 3. Our concluding remarks comprise Section 4.

## 2 The Expansion Rate

In the presence of a cosmological constant  $\Lambda$  the gravitational field equations are: <sup>1</sup>

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0 . \quad (1)$$

The standard local definition of the expansion rate  $\mathcal{H}$  [5]:

$$\mathcal{H}(t, \mathbf{x}) = \frac{1}{3} D^\mu u_\mu(t, \mathbf{x}) , \quad (2)$$

is in terms of the covariant derivative  $D_\mu$  of a timelike 4-velocity field  $u_\mu$ :

$$g^{\mu\nu}(x) u_\mu(x) u_\nu(x) = -1 . \quad (3)$$

An appropriate 4-velocity field can be constructed from a scalar functional  $\Phi$  of the metric satisfying, for all  $x$ , the dynamical equation: <sup>2</sup>

$$\square\Phi[g](x) = \frac{1}{\sqrt{-g}} \partial_\mu [\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi] = 3H , \quad (4)$$

where  $H$  is the Hubble parameter ( $\Lambda = 3H^2$ ). On the initial value surface the scalar  $\Phi$  satisfies:

$$\Phi(t_I, \mathbf{x}) \Big|_{\text{IVS}} = 0 \quad , \quad -g^{\alpha\beta}(t_I, \mathbf{x}) \partial_\alpha \Phi(t_I, \mathbf{x}) \partial_\beta \Phi(t_I, \mathbf{x}) \Big|_{\text{IVS}} = 1 . \quad (5)$$

The resulting 4-velocity field  $V_\mu$  equals:

$$V_\mu[g](x) \equiv + \frac{\partial_\mu \Phi[g](x)}{\sqrt{-g^{\alpha\beta}(x) \partial_\alpha \Phi[g](x) \partial_\beta \Phi[g](x)}} , \quad (6)$$

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<sup>1</sup>Hellenic indices take on spacetime values while Latin indices take on space values. Our metric tensor  $g_{\mu\nu}$  has spacelike signature  $(-+++)$  and our curvature tensor equals  $R^\alpha_{\beta\mu\nu} \equiv \Gamma^\alpha_{\nu\beta,\mu} + \Gamma^\alpha_{\mu\rho} \Gamma^\rho_{\nu\beta} - (\mu \leftrightarrow \nu)$ .

<sup>2</sup>The construction that follows has been described in detail in [4]. Further approaches to invariant expansion observables can be found in [6, 7, 8, 9].

and the expansion variable according to (2) is:

$$\mathcal{H}[g](x) = \frac{1}{3} D^\mu V_\mu[g](x) = \frac{1}{3} \frac{1}{\sqrt{-g}} \partial_\mu [\sqrt{-g} g^{\mu\nu} V_\nu] . \quad (7)$$

We can invariantly fix the observation time by specifying the surfaces of simultaneity as follows:

$$\Phi[g](\vartheta[g](x), \mathbf{x}) = \Phi_{\text{dS}}(t) , \quad (8)$$

where  $\Phi_{\text{dS}}(t)$  is the scalar  $\Phi$  in de Sitter spacetime. This requirement determines the functional  $\vartheta[g](x)$  or, equivalently, the observation time.

Our observable  $H$  – which physically represents the expansion rate of spacetime – is given by :

$$H[g](x) \equiv \mathcal{H}[g](\vartheta[g](x), \mathbf{x}) . \quad (9)$$

Under general coordinate transformations which preserve the initial value surface, the variable just constructed transform thusly:

$$\mathcal{H}[g'](x) = \mathcal{H}[g](x'^{-1}(x)) \quad , \quad H[g'](t, \mathbf{x}) = H[g](t, x'^{-1}(t, \mathbf{x})) . \quad (10)$$

### 3 The Classical Computation on the Initial Value Surface

We now turn to the main results of this study, the calculation of the value and first time derivative of the expansion rate observable on the IVS. Because  $\Phi|_{\text{IVS}} = 0$ , the invariant observation time condition (8) is automatically satisfied on the IVS and need not concern us. Consequently, it suffices to consider the expansion rate as provided by  $\mathcal{H}$ .

- *The 3 + 1 decomposition.*

The nature of our problem suggests that we employ a coordinate system that separates space and time.<sup>3</sup> The 3 + 1 decomposition of the line element is:

$$\begin{aligned} ds^2 &= -g_{00}dt^2 + 2g_{0i} dt dx^i + g_{ij} dx^i dx^j \\ &= -N^2 dt^2 + \gamma_{ij} (dx^i + N^i dt)(dx^j + N^j dt) , \end{aligned} \quad (11)$$

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<sup>3</sup>The pioneering work on the subject by Arnowitt, Deser and Misner (ADM) can be found in [10]; see also [11, 12].

with  $N$  the lapse,  $N^i$  the shift vector and  $\gamma_{ij}$  the spatial metric. It follows that the elements of the the spacetime metric  $g_{\mu\nu}$  are:

$$g_{00} = -N^2 + N_i N^i \quad , \quad g_{0i} = N_i \quad , \quad g_{ij} = \gamma_{ij} \quad , \quad (12)$$

while those of its inverse metric  $g^{\mu\nu}$  are:

$$g^{00} = -\frac{1}{N^2} \quad , \quad g^{0i} = \frac{N^i}{N^2} \quad , \quad g^{ij} = \gamma^{ij} - \frac{N^i N^j}{N^2} \quad . \quad (13)$$

The relevant Christoffel connections are:<sup>4</sup>

$$\Gamma^0_{00} = \frac{N_{,0}}{N} + \frac{N^i N_{,i}}{N} - \frac{N^i N^j N_{;ij}}{N^2} \quad , \quad (14)$$

$$\Gamma^0_{0i} = \frac{N_{,i}}{N} - \frac{N^j K_{ij}}{N} \quad , \quad (15)$$

$$\Gamma^0_{ij} = -\frac{K_{ij}}{N} \quad , \quad (16)$$

$$\begin{aligned} \Gamma^i_{00} = & -\frac{N^i}{N} [N_{,0} + N^j N_{,j} - N^j N^k K_{jk}] + N^i_{,0} - N N^{,i} - 2N N^j K^i_j \\ & + N^j N^i_{;j} \quad , \end{aligned} \quad (17)$$

$$\Gamma^i_{j0} = -\frac{N^i N_{,j}}{N} + N^i_{,j} - \left( \gamma^{ik} - \frac{N^i N^k}{N^2} \right) N K_{kj} \quad , \quad (18)$$

$$\Gamma^i_{jk} = \frac{N^i K_{jk}}{N} + \bar{\Gamma}^i_{jk} \quad , \quad (19)$$

where  $K_{ij}$  is the extrinsic curvature.

The gravitational field equations (1) can be separated into evolution equations and constraints. The former are:

$$\partial_0 \gamma_{ij} = -2N K_{ij} + \bar{D}_i N_j + \bar{D}_j N_i \quad , \quad (20)$$

$$\begin{aligned} \partial_0 K_{ij} = & -\bar{D}_i \bar{D}_j N + N^k \bar{D}_k K_{ij} + K_{ik} \bar{D}_j N^k + K_{jk} \bar{D}_i N^k \\ & + N [\bar{R}_{ij} - 2K_{ik} K^k_j + K K_{ij} - 3H^2 \gamma_{ij}] \quad , \end{aligned} \quad (21)$$

while the latter take the form:

$$\bar{R} + K^2 - K_{ij} K^{ij} = 6H^2 \quad , \quad (22)$$

$$\bar{D}_j (K^{ij} - \gamma^{ij} K) = 0 \quad , \quad (23)$$

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<sup>4</sup>Henceforth, a bar over a symbol indicates it is purely spatial, a comma indicates a derivative with respect to the spatial metric  $\gamma_{ij}$ , and a semicolon a covariant derivative.

where  $K \equiv \gamma^{ij} K_{ij}$  is the trace of the extrinsic curvature and  $\bar{D}_i$  is the spatial covariant derivative with respect to  $\gamma_{ij}$ .

In this decomposition, there are  $12 = 6 + 6$  canonical degrees of freedom that  $\gamma_{ij}$  and  $K_{ij}$  contain. Of these, only  $4 = 2 + 2$  are dynamical and correspond to the two polarization states of the graviton; the other are the 4 constrained degrees of freedom from the 1 + 3 constraint equations (22-23), and the 4 gauge degrees of freedom from the initial coordinate system choices.

- *The elements of the observable on the IVS.*

The equation of motion (4) of the scalar  $\Phi$  is:

$$3H = \square\Phi = g^{\mu\nu} D_\mu D_\nu \Phi = g^{00} D_0 D_0 \Phi + 2g^{0i} D_0 D_i \Phi + g^{ij} D_i D_j \Phi \quad (24)$$

$$= g^{\mu\nu} (\Phi_{,\mu\nu} - \Gamma_{\mu\nu}^\rho \Phi_{,\rho}) . \quad (25)$$

The initial value conditions (5) on the scalar  $\Phi$  can be conveniently written as:

$$\Phi|_{\text{IVS}} = 0 \quad , \quad \Phi_{,\mu}|_{\text{IVS}} = -N \delta_\mu^0 . \quad (26)$$

We shall also need the following initial covariant derivatives of  $\Phi$ :

$$D_0 D_0 \Phi|_{\text{IVS}} = [\Phi_{,00} - \Gamma_{00}^\rho \Phi_{,\rho}]|_{\text{IVS}} = -N^2(3H + K) - N^i N^j K_{ij} , \quad (27)$$

$$D_0 D_i \Phi|_{\text{IVS}} = [\Phi_{,0i} - \Gamma_{0i}^\rho \Phi_{,\rho}]|_{\text{IVS}} = -N^j K_{ji} , \quad (28)$$

$$D_i D_j \Phi|_{\text{IVS}} = [\Phi_{,ij} - \Gamma_{ij}^\rho \Phi_{,\rho}]|_{\text{IVS}} = -K_{ij} , \quad (29)$$

$$D_0 D_i D_j \Phi|_{\text{IVS}} = [(D_i D_j \Phi)_{,0} - \Gamma_{0i}^\rho D_\rho D_j \Phi - \Gamma_{0j}^\rho D_i D_\rho \Phi]|_{\text{IVS}} \quad (30)$$

$$= +3NH^2 \gamma_{ij} - 3HNK_{ij} - 2NKK_{ij} - \frac{1}{N} K_{ij} N^k N^l K_{kl} \\ + \frac{1}{N^2} K_{ij} N^k N^l N_{k;l} - N^k K_{ij;k} - N \bar{R}_{ij} . \quad (31)$$

In view of (26), the 4-velocity field (6) becomes:

$$V_\mu|_{\text{IVS}} = -N \delta_\mu^0 \quad , \quad V^\mu|_{\text{IVS}} = +\frac{1}{N} \delta_0^\mu . \quad (32)$$

- *The observable on the IVS.*

The general form of the local expansion rate is given by (7):

$$\mathcal{H} = \frac{1}{3} D_\mu V^\mu = \frac{1}{3} \frac{1}{\sqrt{-g}} \partial_\mu \left( \frac{\sqrt{-g} g^{\mu\nu} \Phi_{,\nu}}{\sqrt{-g^{\alpha\beta} \Phi_{,\alpha} \Phi_{,\beta}}} \right) \quad (33)$$

$$= \frac{1}{3} \frac{\square\Phi}{\sqrt{-g^{\alpha\beta} \Phi_{,\alpha} \Phi_{,\beta}}} + \frac{g^{\mu\nu} \Phi_{,\mu} g^{\rho\sigma} \Phi_{,\rho} D_\nu D_\sigma \Phi}{3 (-g^{\alpha\beta} \Phi_{,\alpha} \Phi_{,\beta})^{\frac{3}{2}}} . \quad (34)$$

When restricting to the initial value surface we sequentially obtain:

$$\mathcal{H}|_{\text{IVS}} = \frac{1}{3} \left( 3H + g^{\mu\nu} \Phi_{,\mu} g^{\rho\sigma} \Phi_{,\rho} D_\nu D_\sigma \Phi \right) \Big|_{\text{IVS}} \quad (35)$$

$$= \frac{1}{3} \left( 3H + N^2 g^{0\nu} g^{0\sigma} D_\nu D_\sigma \Phi \right) \Big|_{\text{IVS}} \quad (36)$$

$$= \frac{1}{3} \left\{ 3H + N^2 \left[ g^{00} g^{00} D_0 D_0 \Phi + 2g^{00} g^{0i} D_0 D_i \Phi + g^{0i} g^{0j} D_i D_j \Phi \right] \right\} \Big|_{\text{IVS}} \quad (37)$$

$$= \frac{1}{3} \left\{ 3H + N^2 \left[ -N^{-2} \left( 3H - g^{ij} D_i D_j \Phi \right) + g^{0i} g^{0j} D_i D_j \Phi \right] \right\} \Big|_{\text{IVS}} \quad (38)$$

$$= \frac{1}{3} \gamma^{ij} D_i D_j \Phi \Big|_{\text{IVS}} = -\frac{1}{3} \gamma^{ij} K_{ij} \Big|_{\text{IVS}} = -\frac{1}{3} K \Big|_{\text{IVS}} \quad (39)$$

where – besides the form of the metric (13) and the double covariant derivative (29) of  $\Phi$  – we have used the equation of motion (24). Since  $K$  is a pure gauge degree of freedom we conclude that  $\mathcal{H}$  can take *any* initial value of our choice. Therefore, we can make it vanish on the IVS by the gauge choice  $K = 0$  and then ask whether it will stay zero under time evolution.

- *The first time derivative of the observable on the IVS.*

In order to investigate the behaviour of the observable under infinitesimal time evolution, we consider its first derivative:

$$D_\mu \mathcal{H} = \frac{H g^{\kappa\lambda} \Phi_{,\kappa} D_\mu D_\lambda \Phi}{(-g^{\alpha\beta} \Phi_{,\alpha} \Phi_{,\beta})^{\frac{3}{2}}} + \frac{H g^{\kappa\lambda} \Phi_{,\kappa} g^{\rho\sigma} \Phi_{,\rho} (D_\lambda D_\sigma \Phi) g^{\gamma\delta} \Phi_{,\gamma} D_\mu D_\delta \Phi}{(-g^{\alpha\beta} \Phi_{,\alpha} \Phi_{,\beta})^{\frac{5}{2}}} + \frac{\frac{2}{3} g^{\kappa\lambda} \Phi_{,\kappa} g^{\rho\sigma} (D_\mu D_\rho \Phi) D_\lambda D_\sigma \Phi + \frac{1}{3} g^{\kappa\lambda} \Phi_{,\kappa} g^{\rho\sigma} \Phi_{,\rho} D_\mu D_\lambda D_\sigma \Phi}{(-g^{\alpha\beta} \Phi_{,\alpha} \Phi_{,\beta})^{\frac{3}{2}}} , \quad (40)$$

which on the initial value surface the reduces to:

$$D_\mu \mathcal{H} \Big|_{\text{IVS}} = H g^{0\lambda} (-N D_\mu D_\lambda \Phi) + N^2 g^{0\lambda} g^{0\sigma} D_\lambda D_\sigma (-N g^{0\delta} D_\mu D_\delta \Phi) - \frac{2}{3} N g^{0\lambda} g^{\rho\sigma} (D_\mu D_\rho \Phi) D_\lambda D_\sigma \Phi + \frac{1}{3} N^2 g^{0\lambda} g^{0\sigma} D_\mu D_\lambda D_\sigma \Phi . \quad (41)$$

We shall be interested in the  $\mu = 0$  component of (41) that we write schematically as the sum of four terms, respectively corresponding to the four terms of (41):

$$\partial_0 \mathcal{H} \Big|_{\text{IVS}} \equiv I_1 + I_2 + I_3 + I_4 . \quad (42)$$

For their reduction, the derivative of the  $\Phi$  equation of motion (24):

$$g^{00} D_\mu D_0 D_0 \Phi + 2g^{0i} D_\mu D_0 D_i \Phi + g^{ij} D_\mu D_i D_j \Phi = 0 , \quad (43)$$

has been useful:

$$\begin{aligned} I_1 &\equiv Hg^{0\lambda}(-ND_0D_\lambda\Phi) \\ &= -NH[3H - g^{0i}D_0D_i\Phi - g^{ij}D_iD_j\Phi] , \end{aligned} \quad (44)$$

$$\begin{aligned} I_2 &\equiv N^2g^{0\lambda}g^{0\sigma}D_\lambda D_\sigma(-Ng^{0\delta}D_0D_\delta\Phi) \\ &= -N^3\{g^{00}[3H - g^{ij}D_iD_j\Phi] + g^{0i}g^{0j}D_iD_j\Phi\} \\ &\quad \times [3H - g^{0k}D_0D_k\Phi - g^{kl}D_kD_l\Phi] , \end{aligned} \quad (45)$$

$$\begin{aligned} I_3 &\equiv -\frac{2}{3}Ng^{0\lambda}g^{\rho\sigma}(D_0D_\rho\Phi)D_\lambda D_\sigma\Phi \\ &= -\frac{2}{3}N\{9H^2 + 3H[-2g^{ij}D_iD_j\Phi - g^{0i}D_0D_i\Phi + \frac{g^{0i}g^{0j}}{g^{00}}D_iD_j\Phi] \\ &\quad + [-g^{0i}g^{0j} + g^{00}g^{ij}](D_0D_i\Phi)D_0D_j\Phi \\ &\quad + [g^{ij}g^{kl} - \frac{g^{0i}g^{0j}}{g^{00}}g^{kl}](D_iD_j\Phi)D_kD_l\Phi \\ &\quad + [g^{0i}g^{jk} - 2g^{0i}\frac{g^{0j}g^{0k}}{g^{00}} + g^{0j}g^{ik}](D_0D_i\Phi)D_jD_k\Phi\} , \end{aligned} \quad (46)$$

$$\begin{aligned} I_4 &\equiv \frac{1}{3}N^2g^{0\lambda}g^{0\sigma}D_0D_\lambda D_\sigma\Phi \\ &= \frac{1}{3}N^2[-g^{00}g^{ij} + g^{0i}g^{0j}]D_0D_iD_j\Phi . \end{aligned} \quad (47)$$

Grouping together the terms from expansions (44-47) according to their  $H$  content we notice that:

- (i) the terms proportional to  $H^2$  cancel when added up,
- (ii) the terms proportional to  $H$  add up to,

$$J_1 \equiv NH[-g^{ij} - N^2g^{0i}g^{0j}]D_iD_j\Phi = -NH\gamma^{ij}D_iD_j\Phi = +NHK , \quad (48)$$

(iii) the terms without  $H$  dependence – and organized according to their covariant derivatives structure – are,

$$J_2 \equiv N(D_0D_i\Phi)D_0D_j\Phi \times \frac{2}{3N^2}[g^{ij} + N^2g^{0i}g^{0j}] \quad (49)$$

$$= +\frac{2}{3N}\gamma^{ij}(D_0D_i\Phi)D_0D_j\Phi = +\frac{2}{3N}N^iN^jK_i^kK_{kj} , \quad (50)$$

$$J_{3a} \equiv N(D_0D_i\Phi)D_jD_k\Phi \times \frac{1}{3}g^{0i}[g^{jk} + N^2g^{0j}g^{0k}] \quad (51)$$



$$= +\frac{1}{3N}N^i\gamma^{jk}(D_0D_i\Phi)D_jD_k\Phi = +\frac{1}{3N}KN^iN^jK_{ij} \ , \quad (52)$$

$$J_{3b} \equiv N(D_0D_i\Phi)D_jD_k\Phi \times \frac{2}{3}g^{0j}\left[-g^{ik}-N^2g^{0i}g^{0k}\right] \quad (53)$$

$$= -\frac{2}{3N}N^j\gamma^{ik}(D_0D_i\Phi)D_jD_k\Phi = -\frac{2}{3N}N^iN^jK_i^kK_{kj} \ , \quad (54)$$

$$J_4 \equiv N(D_iD_j\Phi)D_kD_l\Phi \times \frac{1}{3}g^{ij}\left[g^{kl}+N^2g^{0k}g^{0l}\right] \quad (55)$$

$$\begin{aligned} &= +\frac{1}{3}N\left[\gamma^{ij}-\frac{N^iN^j}{N^2}\right]\gamma^{kl}(D_iD_j\Phi)D_kD_l\Phi \\ &= +\frac{1}{3}NK^2 - \frac{1}{3N}KN^iN^jK_{ij} \ , \end{aligned} \quad (56)$$

$$J_5 \equiv D_0D_iD_j\Phi \times \frac{1}{3}\left[g^{ij}+N^2g^{0i}g^{0j}\right] \quad (57)$$

$$\begin{aligned} &= +\frac{1}{3}\gamma^{ij}D_0D_iD_j\Phi = +3NH^2 - \frac{1}{3}\bar{R} - NHK - \frac{2}{3}NK^2 \\ &\quad -\frac{1}{3}N^iK_{,i} - \frac{1}{3N}KN^iN^jK_{ij} + \frac{1}{3N^2}KN^iN^jN_{i;j} \ . \end{aligned} \quad (58)$$

The final result is the sum of the above (ii) and (iii) terms:

$$\partial_0\mathcal{H}\Big|_{\text{IVS}} = J_1 + J_2 + J_{3a} + J_{3b} + J_4 + J_5 \quad (59)$$

$$\begin{aligned} &= 3NH^2 - \frac{1}{3}N\bar{R} - \frac{N}{3}K^2 - \frac{1}{3}N^iK_{,i} - \frac{1}{3N}KN^iN^jK_{ij} \\ &\quad + \frac{1}{3N^2}KN^iN^jN_{i;j} \ . \end{aligned} \quad (60)$$

We use our gauge freedom to impose  $K\Big|_{\text{IVS}} = 0$  as the gauge condition so that (60) becomes:

$$\partial_0\mathcal{H}\Big|_{\text{IVS}} = N\left(3H^2 - \frac{1}{3}\bar{R}\right) \ . \quad (61)$$

Furthermore, the constraint equation (22) in  $K\Big|_{\text{IVS}} = 0$  gauge is:

$$\bar{R} = 6H^2 + K_{ij}K^{ij} \ , \quad (62)$$

implying finally:

$$\partial_0\mathcal{H}\Big|_{\text{IVS}} = N\left(H^2 - \frac{1}{3}K_{ij}K^{ij}\right) \ . \quad (63)$$

The lapse function  $N$  sets the choice of physical time as opposed to the coordinate time  $t$ . Because  $K_{ij}K^{ij}$  is positive we conclude that the expansion rate can indeed diminish. The presence of the diminishing term for any value of  $H$  indicates that it has the ability to completely cancel  $H^2$ .

- *The correspondence limits.*

A minimum requirement for our results is to be consistent with various correspondence limits. Of particular interest is the case of de Sitter spacetime. When we consider the open coordinate system – the cosmological patch – we have:

$$N = 1 \quad , \quad N^i = 0 \quad , \quad \gamma_{ij} = e^{2Ht} \delta_{ij} \quad , \quad (64)$$

so that – from (20) – we obtain:

$$K_{ij} = -H\gamma_{ij} \quad , \quad K = -3H \quad , \quad (65)$$

implying:

$$\mathcal{H}|_{\text{IVS}} = H \quad , \quad \partial_0 \mathcal{H}|_{\text{IVS}} = 0 \quad . \quad (66)$$

The expansion rate started at  $H$  and stays at  $H$ .

When we consider the closed coordinate system – the full manifold – we have:

$$N = 1 \quad , \quad N^i = 0 \quad , \quad \gamma_{ij} = H^{-2} \cosh^2(H\tau) \Omega_{ij} \quad , \quad (67)$$

where  $\Omega_{ij}$  is the angular line element. Therefore – using (20) – we get:

$$K_{ij} = -H \tanh(H\tau) \gamma_{ij} \quad . \quad (68)$$

The choice of  $\tau = 0$  as the initial value surface – corresponding to the throat of the hyperboloid – implies that  $K_{ij}|_{\text{IVS}} = 0$  and we conclude that the system started with no expansion and instantaneously began accelerating:

$$\mathcal{H}|_{\text{IVS}} = 0 \quad , \quad \partial_0 \mathcal{H}|_{\text{IVS}} = H^2 \quad . \quad (69)$$

The  $\Lambda = 3H^2 = 0$  limit gives:

$$\mathcal{H}|_{\text{IVS}} = 0 \quad , \quad \partial_0 \mathcal{H}|_{\text{IVS}} = -\frac{N}{3} K_{ij} K^{ij} \quad , \quad (70)$$

leading to contraction when  $K_{ij} \neq 0$ .

Finally, in the flat spacetime limit:

$$N = 1 \quad , \quad N^i = 0 \quad , \quad \gamma_{ij} = \delta_{ij} \quad , \quad (71)$$

the expansion rate  $\mathcal{H}$  vanishes for all time.

## 4 Epilogue

On the initial value surface the expansion rate observable is proportional to the trace  $K$  of the extrinsic curvature, which is a gauge degree of freedom. It follows that the natural gauge choice is  $K = 0$  because it allows us to start with zero expansion rate and let time evolution determine what follows. What we found – and this is the main physical message of our non-perturbative classical computation – is that there exist initial value data corresponding to configurations with  $K_{ij} \neq 0$  which reduce the expansion rate in the presence of a cosmological constant  $\Lambda$ . Furthermore, there seems to be no obstacle for that reduction to completely arrest the initial expansion due to  $\Lambda = 3H^2$ .

Indeed from the constraint equation of motion (23) and in  $K = 0$  gauge, the only requirement on the initial extrinsic curvature is that its covariant divergence vanishes. Moreover, the dimensionality of  $K_{ij}$  is that of mass and there are only two mass scales at our disposal: the inflationary scale  $H$  and the Planck scale  $M_{\text{Pl}}$ . An upper bound on  $K_{ij}K^{ij}$  cannot vanish with  $H$  vanishing because there exist configurations with  $H = 0$  &  $K_{ij} > 0$  in direct contradiction. Thus, any upper bound – if one exists – must involve  $M_{\text{Pl}}$ , a situation which still allows cancellation of the  $H^2$  term in (63) because  $M_{\text{Pl}}^2 \gg H^2$ .

It is worth noting that our results confirm again the fact that the equation of motion  $R = 4\Lambda$  cannot determine the physical expansion rate. This was an issue debated in [14] and our present classical calculation – in which *every* configuration obeys the equation  $R = 4\Lambda$  – shows that *any* configuration with  $K_{ij} \neq 0$  is held together at least infinitesimally and hence deviates from the de Sitter expansion.

There are many cases in which self-gravitation drastically alters the properties of a physical system. For instance, it can eliminate the bare mass of a point particle [10], or cause gravitational collapse in a system of incoming gravity waves [13]. The underlying mechanism can be seen in the Hamiltonian constraint (22) as the interplay between the kinetic energy term –  $K_{ij}K^{ij}$  – and the potential energy term – the non-linear parts buried in  $\bar{R}$ .

For the physical situation at hand, once inflationary gravitons are produced their effect on cosmological evolution can be understood in completely classical terms. Since gravitational waves attract each other and act to diminish expansion, when enough of them are present they *can* completely stop it and even reverse the trend leading to collapse. It should be possible to find

a classical configuration of gravitational waves such that the universe holds itself together, against the tendency for de Sitter expansion. Such a classical state will almost certainly *not* be completely stable but if it is formed from the steady production of infrared gravitons over a prolonged period of inflation, by causality the decay time would almost certainly be *longer* than the lifetime of the universe.

We do not know what initial value data describe this classical configuration of gravitons. We do however know from our present analysis that initial value data *exist* for which the corresponding configuration does not succumb to accelerated expansion. It would be very significant to explicitly verify that inflationary graviton production eventually forms a state of the kind that stops inflation. Of course even if the latter is not completely the case, the very existence of such a configuration implies that there is some probability for the universe to tunnel to it and, hence, stop inflation.

Finally, we itemize our main conclusions. In the presence of a cosmological constant:

- (i) The initial value of the expansion rate can be gauged to zero;
- (ii) The presence of initial gravitational waves with  $K_{ij} \neq 0$  makes the initial time derivative of the expansion rate less than its value in de Sitter;
- (iii) It seems that nothing precludes initial value data which make the initial first derivative of the expansion rate vanish; and
- (iv) The evolution of the universe is a *sustained gravitational collapse*.

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